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Algebraic Bethe ansatz for the non-linear Schrödinger model: II. Mixed fermion and boson fields

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Abstract. The Bethe ansatz equations for a mixture of species of fermions and bosons in one dimension, interacting with a repulsive delta potential, are obtained. It is shown that the matrix periodic boundary conditions of the system can be obtained from those of a fermion or boson system. Finally, the extension of these results to the non-linear Schrödinger model with a supermatrix is made.

1. Introduction

We consider a one-dimensional N-body system with a repulsive delta potential whose Hamiltonian is

$$H = \int \mathrm{d}x[(\mathrm{d}q^+/\mathrm{d}x)(\mathrm{d}q/\mathrm{d}x) + c{:}q^+qq^+q{:}] \tag{1}$$

where q is a supermatrix and has the structure $(m, n)\times(k, 0)$, i.e. $q\sim(m, n)\times(k, 0)$. However, there exist some particularly interesting cases: the model (1) with $q\sim(1,0)\times(1,0)$ was studied by Lieb and Liniger (1963), Faddeev (1981) and Thacker (1981), the model with $q\sim(2,0)\times(0,1)$ was studied by Yang (1967), and the one with $q\sim(m\geqslant 2,0)\times(0,1)$ was studied by Sutherland (1968) and Zhou and Zhao (1986). Yang reduced the problem to a matrix problem and applied the second Bethe ansatz (BA) to derive the BA equations. The matrix periodic boundary conditions (PBC) for the model (1) with $q\sim(m,0)\times(1,0)$ and $(0,m)=(1,0)$ were obtained and the relationship between them was discussed by Zhou (1988). The model (1) with $q\sim(1,2)\times(1,0)$, i.e. a mixture of two species of fermions and one species of bosons, was studied by Lai and Yang (1971) and that with the general case $q\sim(m,n)\times(k,0)$ was studied by Fan *et al* (1986). However, the BA equations derived by Fan *et al* (1986) cannot be directly reduced to those in Lai and Yang's work (1971) as $m=1$, $n=2$ and $k=1$. Both $q\sim(m,n)\times(1,0)$ and $(n,m)\times(0,1)$ represent a mixture of m species of bosons and n species of fermions. The purpose of this paper, therefore, is to derive the BA equations given by Lai and Yang (1971) and further derive the BA equations for $q\sim(m,n)\times(1,0)$, $q\sim(n,m)\times(1,0)$ and $q\sim(m,n)\times(k,0)$ by using the quantum inverse scattering method (QISM).

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2. The matrix PBC

In this paper, we approach the problem by studying the model (1) with $q\sim(m+n,0)\times(1,0)$ and $(0,m+n)\times(1,0)$ instead of $q\sim(m,n)\times(1,0)$ and $(n,m)\times(1,0)$ respectively from the start. $q\sim(m+n,0)\times(1,0)$ and $(0,m+n)\times(1,0)$ represent a system having $m+n$ species of bosons ($p=1$) and fermions ($p=-1$) respectively. The matrix PBC are (Zhou 1988)

$$\begin{aligned} &t(\lambda_j,p)_{m+n}f_{m+n}=\nu(\lambda_j,p,R_1)f_{m+n}\\ &t(\lambda,p)_{m+n}=\mathrm{Tr}(L(\lambda-\lambda_N)_{m+n}\ldots L(\lambda-\lambda_1)_{m+n})\\ &L(\lambda-\lambda_j)_{m+n}=\alpha(\lambda-\lambda_j)+\beta(\lambda-\lambda_j)P^j_{m+n}\\ &\beta(\lambda-\lambda_j)=1-\alpha(\lambda-\lambda_j)=(-\mathrm{i}pc)/(\lambda-\lambda_j-\mathrm{i}pc) \end{aligned}\tag{2}$$

where $L(\lambda-\lambda_j)_{m+n}$ can be considered as the transfer matrix at site j of a generalised Heisenberg ferromagnetic chain with dynamical variable $e_{ab}(j)$ and auxiliary variable $e_{ba}(m+n)$. The permutation operator P^j_{m+n} is

$$P^j_{m+n}=e_{ab}(j)e_{ba}(m+n)$$

where the double indices a and b mean summations over $1,2,\ldots,m+n$. $(e_{ab})_{cd}=\delta_{ac}\delta_{bd}$. The eigenvalue $\nu(\lambda_j,p,R_1)$ is

$$\nu(\lambda_j,p,R_1)=\exp(-2\mathrm{i}\lambda_jL)\prod_{i\neq j}^{N}\frac{\lambda_j-\lambda_i+\mathrm{i}c}{\lambda_j-\lambda_i-\mathrm{i}pc}\tag{3}$$

where the λ are the momenta of N particles, $p=1$ for bosons and $p=-1$ for fermions. For a system having $m+n$ species of particles with the particle number content $N-M_1$, $M_1-M_2,\ldots,M_{m+n-2}-M_{m+n-1}$, M_{m+n-1}, equation (2) has the symmetry $R_1=[N-M_1,M_1-M_2,\ldots,M_{m+n-1}]$, which is an irreducible representation of the permutation group S_N.

By using QISM equation (2) can be solved to derive the BA equations for fermions $p=-1$ or bosons $p=1$ (Zhou 1988). Here equation (2), however, is first reduced to another matrix equation of smaller dimension for $R_2=[M_m-M_{m+1},\ldots,M_{m+n-1}]$

$$\begin{aligned} &t(\lambda_j^{(m)},p)_nf_n=\nu(\lambda_j^{(m)},p,R_2)_nf_n\\ &t(\lambda^{(m)},p)_n=\mathrm{Tr}(L(\lambda^{(m)}-\lambda^{(m)}_{M_m})_n\ldots L(\lambda^{(m)}-\lambda_1^{(m)})_n) \end{aligned}\tag{4}$$

where $L(\lambda^{(m)}-\lambda_j^{(m)})_n$ is $L(\lambda-\lambda_j)_{m+n}$ with P^j_n and $\lambda^{(m)}-\lambda_j^{(m)}$ replacing P^j_{m+n} and $\lambda-\lambda_j$ respectively, and P^j_n is given by

$$P^j_n=e_{cd}(h)\,e_{dc}(n)$$

where the double indices d and c mean summations over $1,2,\ldots,n$. Equation (4) is identical in form to the original equation (2). However, equation (4) is that for the system having n species of particles with the particle number content $M_m-M_{m+1},\ldots,M_{m+n-1}$. The relationship between equation (2) for bosons $p=1$ and that for fermions $p=-1$ with the same species content has been discussed by Zhou (1988) and we have

$$\nu(\lambda_j^{(m)},-p,R_2)_n=\nu(\lambda_j^{(m)},p,R_2^*)_n\prod_{i\neq j}^{M_m}\frac{\lambda_j^{(m)}-\lambda_i^{(m)}-\mathrm{i}pc}{\lambda_j^{(m)}-\lambda_i^{(m)}+\mathrm{i}pc}\tag{5}$$

where R_2^* is a conjugate representation of R_2. If R_2 represents a boson system with n species of particles then R_2^* describes a fermion system with the same species content and vice versa. Hence we may consider that the $\nu(\lambda_j^{(m)}, p, R_2)$ is the eigenvalue of the equation (4) for p-particles and $\nu(\lambda_j^{(m)}, p, R_2^*)$ is that for $-p$-particles where the p-particles ($-p$-particles) represent the bosons (fermions) and the fermions (bosons) for $p=1$ and -1 respectively. Using equation (5) equation (4) for R_2^* can be written as

$$t(\lambda_j^{(m)}, -p)_n f_n = \nu(\lambda_j^{(m)}, -p, R_2)_n f_n. \tag{6}$$

Equation (6) is equation (4) with $-p$ replacing p and can be solved also by using the QISM. The final results obtained are those for a mixture system having m species of p-particles with the particle number content $N-M_1, \ldots, M_{m-1}-M_n$ and n species of $-p$-particles with the particle number content $M_m - M_{m+1}, \ldots, M_{m+n-1}$.

Identically we may solve the equation (2) from the start by imposing on f_{m+n} the symmetry $R=[N-M_1, \ldots, M_{m-1}-M_m, R_2^*]$ and $R_2=[M_m-M_{m+1}, \ldots, M_{m+n-1}]$ as shown in figure 1. The condition $n \leqslant M_{m-1}-M_m$, however, must be held.

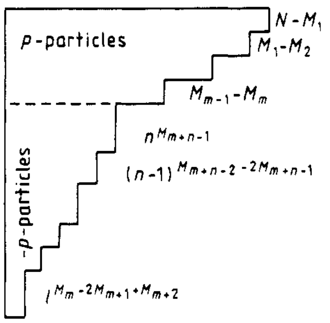

Figure 1. Diagram of the tableau represented by R. The p-particles ($-p$-particles) represent bosons (fermions) and fermions (bosons) for $p=1$ and $p=-1$ respectively.

3. The BA equations for $q \sim (1,2)\times(1,0)$

Lai and Yang considered a one-dimensional N-body QNSM for a mixture system with m_1 fermions of species 1, m_2 fermions of species 2 and m_b bosons, where $N=m_1+m_2+m_b$ and $p=-1$, $n=1$, $N-M_1=m_1$, $M_1-M_2=m_2$ and $M_2=m_b$ according to the discussion in the last section. The BA equations for the mixture system can be obtained by solving equation (2) for $R=[m_1, m_2, 1^m]$. The calculation is straightforward and the results are

$$\exp(2\mathrm{i}\lambda_j L) = \prod_i^{M_1} \alpha(\lambda_i^{(1)}-\lambda_j) \qquad j=1,2,\ldots,N.$$

$$\prod_i^{N} \alpha(\lambda_j^{(1)}-\lambda_i) = -\prod_i^{M_1} \frac{\alpha(\lambda_j^{(1)}-\lambda_i^{(1)})}{\alpha(\lambda_i^{(1)}-\lambda_j^{(1)})} \prod_i^{M_2} \alpha(\lambda_i^{(2)}-\lambda_j^{(1)}) \qquad j=1,2,\ldots,M_1.$$

$$\prod_i^{M_1} \alpha(\lambda_j^{(2)}-\lambda_i^{(1)}) = 1 \qquad j=1,2,\ldots,M_2.$$

Substituting $\lambda_j^{(1)} = \Lambda_j - \frac{1}{2}\mathrm{i}c$ and $\lambda_j^{(2)} = A_j - \mathrm{i}c$ into the BA equations, we will obtain the results of Lai and Yang (1971).

4. The BA equations for $q\sim(m,n)\times(1,0)$ and $(n,m)\times(1,0)$

By using the QISM the equation (2) for R shown in figure 1 can be solved to derive the BA equations. Similar calculations can be found in the papers given by Kulish and Reshetikhin (1981) and Tsvelick and Wiegmann (1983). It is straightforward but rather lengthy. So we write only the final results in the following:

$$\exp(2i\lambda_jL)=\prod_{i\neq j}^{N}\frac{\lambda_j-\lambda_i+ic}{\lambda_j-\lambda_i-ipc}\prod_{i}^{M_1}\alpha(\lambda_i^{(1)}-\lambda_j)\qquad j=1,2,\ldots,N \tag{7}$$

$$\prod_{i}^{N}\alpha(\lambda_j^{(1)}-\lambda_i)=-\prod_{i}^{M_1}\frac{\alpha(\lambda_j^{(1)}-\lambda_i^{(1)})}{\alpha(\lambda_i^{(1)}-\lambda_j^{(1)})}\prod_{i}^{M_2}\alpha(\lambda_i^{(2)}-\lambda_j^{(1)})\qquad j=1,2,\ldots,M_1 \tag{8}$$

$$\prod_{i}^{M_{r-1}}\alpha(\lambda_j^{(r)}-\lambda_i^{(r-1)})=-\prod_{i}^{M_r}\frac{\alpha(\lambda_j^{(r)}-\lambda_i^{(r)})}{\alpha(\lambda_i^{(r)}-\lambda_j^{(r)})}\prod_{i}^{M_{r+1}}\alpha(\lambda_i^{(r+1)}-\lambda_j^{(r)})$$

$$j=1,2,\ldots,M_r\qquad r=2,3,\ldots,m=1 \tag{9}$$

$$\prod_{i}^{M_{m-1}}\alpha(\lambda_j^{(m)}-\lambda_i^{(m-1)})=\prod_{i}^{M_{m+1}}\alpha(\lambda_j^{(m)}-\lambda_i^{(m+1)})\qquad j=1,2,\ldots,M_m \tag{10}$$

$$\prod_{i}^{M_{s-1}}\alpha(\lambda_i^{(s-1)}-\lambda_j^{(s)})=-\prod_{i}^{M_2}\frac{\alpha(\lambda_i^{(s)}-\lambda_j^{(s)})}{\alpha(\lambda_j^{(s)}-\lambda_i^{(s)})}\prod_{i}^{M_{s+1}}\alpha(\lambda_j^{(s)}-\lambda_i^{(s+1)})$$

$$j=1,2,\ldots,M_s\qquad s=m+1,\ldots,m+n-2 \tag{11}$$

$$\prod_{i}^{M_{m+n-2}}\alpha(\lambda_i^{(m+n-2)}-\lambda_j^{(m+n-1)})=-\prod_{i}^{M_{m+n-1}}\frac{\alpha(\lambda_i^{(m+n-1)}-\lambda_j^{(m+n-1)})}{\alpha(\lambda_j^{(m+n-1)}-\lambda_i^{(m+n-1)})}$$

$$j=1,2,\ldots,M_{m+n-1} \tag{12}$$

where $n\leqslant M_{m-1}-M_m$. As $p=-1$ $(=1)$ these are the BA equations for the mixture system having m species of fermions (bosons) with the particle number content $N-M_1$, $M_1-M_2,\ldots,M_{m-1}-M_m$ and n species of bosons (fermions) with the particle number content $M_m-M_{m+1},\ldots,M_{m+n-2}-M_{m+n-1},M_{m+n-1}$, i.e. for the QNSM with $q\sim(n,m)\times(1,0)$ $(q\sim(m,n)\times(1,0))$.

From the BA equations (7)-(12) one may go to the limit N, M, $L\to\infty$ proportionally and discuss the ground-state energy of the system. Details will be published elsewhere.

For $q\sim(1,0)\times(1,0)$, $(0,2)\times(1,0)$, $(0,m\geqslant 2)\times(1,0)$ and $(1,2)\times(1,0)$ our results (7)-(12) coincide with those given by Lieb and Liniger (1963), Yang (1967), Sutherland (1968) and Lai and Yang (1971). For $q\sim(m\geqslant 2,0)\times(1,0)$ the results (7)-(12) coincide with those given by Zhou (1988). Moreover, our method can be applied to the study of the QNSM with $q\sim(m,n)\times(k,0)$.

5. The BA equations for $q\sim(m,n)\times(k,0)$ and $(n,m)\times(k,0)$

The QNSM with $q\sim(m,n)\times(k,0)$ and $(n,m)\times(k,0)$ have the form

$$H=\mathrm{Tr}\int \mathrm{d}x[(\mathrm{d}q^+/\mathrm{d}x)(\mathrm{d}q/\mathrm{d}x)+c{:}q^+qq^+q{:}]. \tag{13}$$

In this section, we will generalise the matrix PBC (2) and (3) to that for $q\sim(m,n)\times(k,0)$ and $(n,m)\times(k,0)$ and derive the BA equations for the model (13). First, the matrix PBC for $q \sim (m+n, 0)\times(k, 0)$ and $(0, m+n)\times(k, 0)$ can be easily obtained by using the QISM and the results are

$$\nu(\lambda_j, p, R_1)\tilde{\nu}(\lambda_j, R')^{-1} = \exp(-2\mathrm{i}\lambda_j L)\prod_{i\neq j}^{N}\frac{\lambda_j-\lambda_i+\mathrm{i}c}{\lambda_j-\lambda_i-\mathrm{i}pc} \tag{14}$$

where $\tilde{\nu}(\lambda_j, R')$ is the eigenvalue of the following equation for $R' = [N-N_1, N_1-N_2, \ldots, N_{k-1}]$:

$$\tilde{t}(\lambda_j)_k g_k = \tilde{\nu}(\lambda_j, R')g_k \tag{15}$$

where

$$\tilde{t}(\lambda)_k = \mathrm{Tr}(L(\lambda_N-\lambda)_k \ldots L(\lambda_1-\lambda)_k)$$

$$L(\lambda_j-\lambda) = a(\lambda_j-\lambda)+b(\lambda_j-\lambda)P_k^j$$

$$b(\lambda_j-\lambda) = 1-a(\lambda_j-\lambda) = (-\mathrm{i}c)/(\lambda_j-\lambda-\mathrm{i}c).$$

$\nu(\lambda_j, p, R_1)$ is given by equation (2) and $p=1$ for $q\sim(m+n, 0)\times(k, 0)$ and $p=-1$ for $q\sim(0, m+n)\times(k, 0)$. According to the discussion in § 2, we can easily generalise the matrix PBC (14) to that with $p=1$ for $q\sim(m, n)\times(k, 0)$ and $p=-1$ for $q\sim(n, m)\times(k, 0)$, i.e. $\nu(\lambda_j, p)$ is given by equation (2) for R shown in figure 1 instead of $R_1 = [N-M_1, \ldots, M_{m+n-2}-M_{m+n-1}, M_{m+n-1}]$. So the BA equations are

$$\exp(2\mathrm{i}\lambda_j L) = \prod_{i\neq j}^{N}\frac{\lambda_j-\lambda_i+\mathrm{i}c}{\lambda_j-\lambda_i-\mathrm{i}pc}\prod_{i}^{M_1}\alpha(\lambda_i^{(1)}-\lambda_j)\prod_{i}^{N_1}a(\lambda_j-\mu_i^{(1)})^{-1} \qquad j=1, 2, \ldots, N \tag{16}$$

$$\prod_{i}^{N}a(\lambda_i-\mu_j^{(1)}) = -\prod_{i}^{N_1}\frac{a(\mu_i^{(1)}-\mu_j^{(1)})}{a(\mu_j^{(1)}-\mu_i^{(1)})}\prod_{i}^{N_2}a(\mu_j^{(1)}-\mu_i^{(2)}) \qquad j=1, 2, \ldots, N_1 \tag{17}$$

$$\prod_{i}^{N_{t-1}}a(\mu_i^{(t-1)}-\mu_j^{(t)}) = -\prod_{i}^{N_t}\frac{a(\mu_i^{(t)}-\mu_j^{(t)})}{a(\mu_j^{(t)}-\mu_i^{(t)})}\prod_{i}^{N_{t+1}}a(\mu_j^{(t)}-\mu_i^{(t+1)})$$

$$j=1, 2, \ldots, N_t \qquad t=2, 3, \ldots, k-2 \tag{18}$$

$$\prod_{i}^{N_{k-2}}a(\mu_i^{(k-2)}-\mu_j^{(k-1)}) = -\prod_{i}^{N_{t-1}}\frac{a(\mu_i^{(k-1)}-\mu_j^{(k-1)})}{a(\mu_j^{(k-1)}-\mu_i^{(k-1)})} \qquad j=1, 2, \ldots, N_{k-1} \tag{19}$$

and the equations (8)-(12).

For a special case with $p=1$ (see figure 1) our results coincide with those in the case studied by Fan *et al* (1986). The BA equations for the QNSM with $q\sim(1, 2)\times(1, 0)$ given by Lai and Yang (1971), however, are included in the results (8)-(12) and (16)-(19) with $p=-1$.

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